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The intermittent small-scale structure of turbulence: data-processing hazards

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Velocity derivatives in turbulence at large Reynolds numbers exhibit probability densities with a large kurtosis. If the value of the kurtosis is about 40, as in many measurements of atmospheric turbulence, careful attention to the processing of such signals is required in order to obtain an acceptable level of accuracy. Signal-to-noise and integration-time limitations make measurements of moments higher than the fourth nearly impossible for signals of this kind.

1. Introduction

A decade ago, Kolmogorov proposed his 'third hypothesis' on the small-scale structure of turbulence (Kolmogorov 1962). This theory, later formalized by Gurvich & Yaglom (1967), is based on the observed intermittency of the dissipation rate in turbulent flows at large Reynolds numbers; it prompted a great amount of experimental research on the probability distribution of turbulent velocity derivatives in atmospheric and oceanic turbulence (e.g. Gibson, Stegen & Williams 1970; Stewart, Wilson & Burling 1970; Wyngaard & Tennekes 1970; Sheih, Tennekes & Lumley 1971). One of the major parameters of interest is the kurtosis (K) of velocity derivatives; in flows with microscale Reynolds numbers (R_λ) of the order of 5000 the kurtosis can reach values as high as 40 (Wyngaard & Tennekes 1970).

Verification of Kolmogorov's third hypothesis requires accurate measurements of K and of the higher moments of the probability density of velocity derivatives. It is the purpose of this paper to show that the present state of the art of data acquisition and processing makes the quest for experimental data on the higher moments next to impossible. The problem is illustrated in figures 1 and 2. Figure 1 is based on atmospheric data obtained during the 1968 Kansas expedition of the Air Force Cambridge Research Laboratories (Wyngaard & Coté 1971); figure 2 is based on laboratory data obtained in 1966 (Wyngaard & Tennekes 1970). The non-dimensionalization employed is as follows. The velocity-derivative signal $\partial u/\partial t$ is called s and has zero mean and a standard deviation σ. The non-dimensional signal x is equal to s/σ and the normalized probability density $\beta_*(x) = \sigma\beta(s)$ satisfies

$$\int_{-\infty}^{\infty} \beta_*(x)\,dx = 1. \quad (1)$$

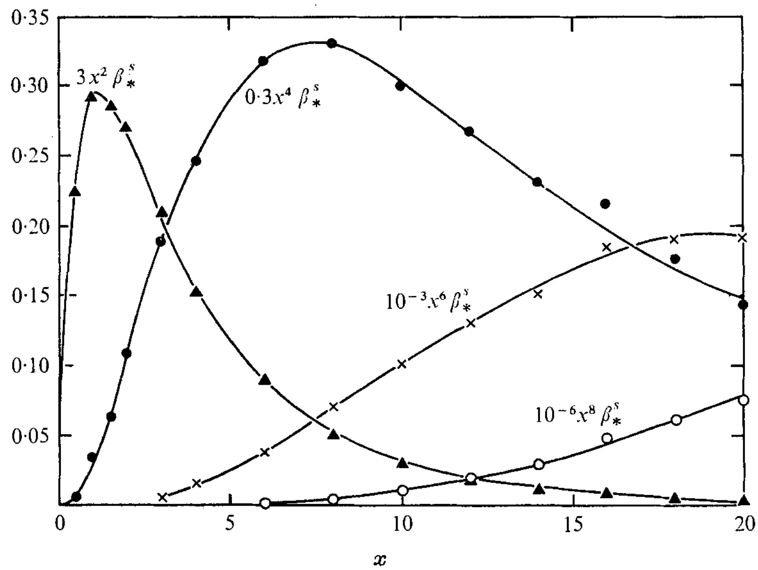

FIGURE 1. Distributions of the second, fourth, sixth and eighth moments of an experimentally obtained probability distribution with $K \simeq 40$. Data from the 1968 Kansas expedition of AFCRL.

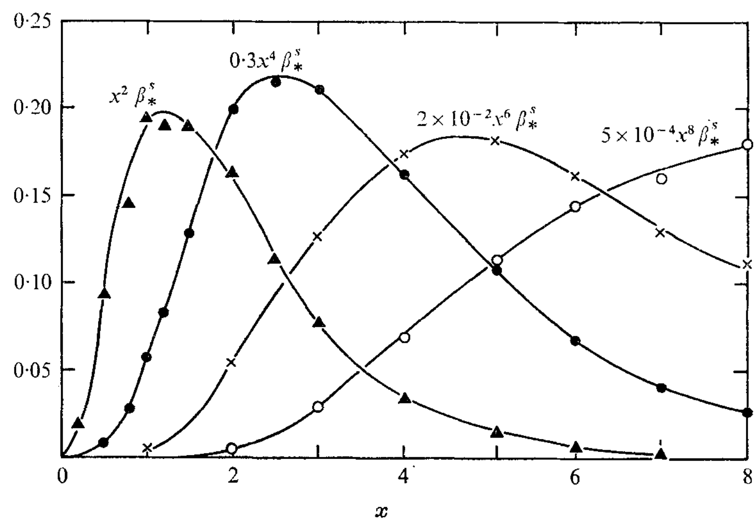

FIGURE 2. Distributions of the second, fourth, sixth and eighth moments of a probability distribution with $K \simeq 6$ obtained in the laboratory.

Since we are mainly interested in even moments, it is convenient to define the symmetric part of β_* by

$$\beta_*^s \equiv \tfrac{1}{2}\{\beta_*(x)+\beta_*(-x)\}, \tag{2}$$

with $0 < x < \infty$. The normalized moment of order $2n$ (n being an integer) then may be written as

$$M_{2n} \equiv \int_{-\infty}^{\infty} x^{2n}\beta_*\,dx = 2\int_0^{\infty} x^{2n}\beta_*^s\,dx. \tag{3}$$

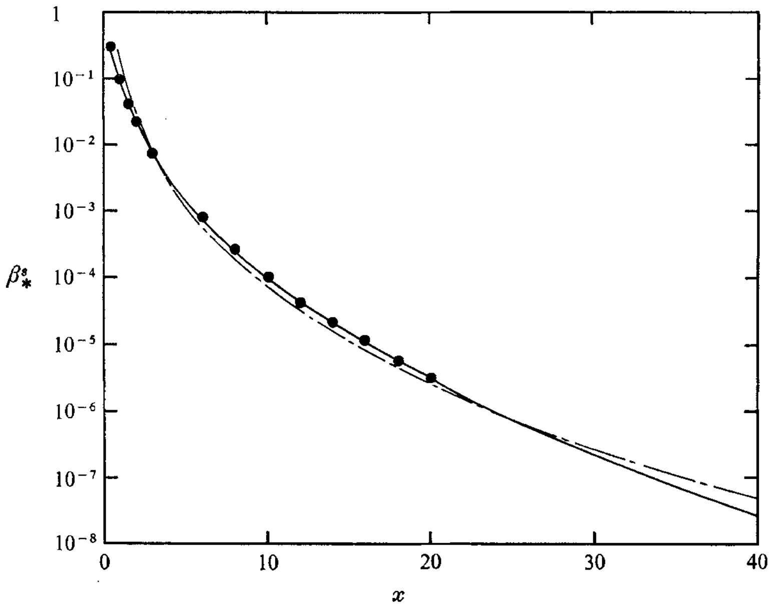

FIGURE 3. The probability density (symmetric part) of velocity derivatives measured in Kansas ($K \cong 40$, $R_\lambda \cong 8000$). ——, curve fit according to (5); — — —, probability distribution with $K = 40$ derived from a log-normal distribution of the squared signal.

In particular, the kurtosis K is given by

$$K \equiv M_4 = 2\int_0^\infty x^4 \beta_*^s \, dx. \tag{4}$$

Figures 1 and 2 contain plots of the integrands $x^{2n}\,\beta_*^s$ as functions of x for $2n = 2$ (the distribution of the variance), $2n = 4$ (the distribution of the kurtosis), $2n = 6$ and $2n = 8$. For the atmospheric data (figure 1) $K = 40$ and $R_\lambda = 8000$ approximately; for the laboratory data (figure 2) $K = 6$ and $R_\lambda = 200$ (Wyngaard & Tennekes 1970). It is evident from figure 1 that, with data available over a range of 20σ on either side of the mean, not even the entire distribution of the kurtosis can be covered if $K = 40$. The situation obviously is worse for the distributions of moments beyond the fourth. Figure 2 shows that signals with a moderate kurtosis do not require the registration of signal excursions beyond $x = 8$ if no data on moments beyond the fourth are required; indeed, the contrast between figures 1 and 2 indicates that the problem of obtaining accurate data is much more severe for measurements at high Reynolds numbers.

The major reason for presenting figures 1 and 2 is to emphasize a crucial, though often overlooked, point: data on the moments of a signal cannot be trusted if at the largest values of x measured the corresponding integrands ($x^{2n}\beta_*^s$) behave erratically or if the integrand has not yet decreased to a level at which the area under the curve can be calculated with reasonable accuracy. An example of the first problem is the occurrence of spurious data points at large values of x, which may go undetected in data processing; the second problem is almost always associated with the finite dynamic range of the instruments employed.

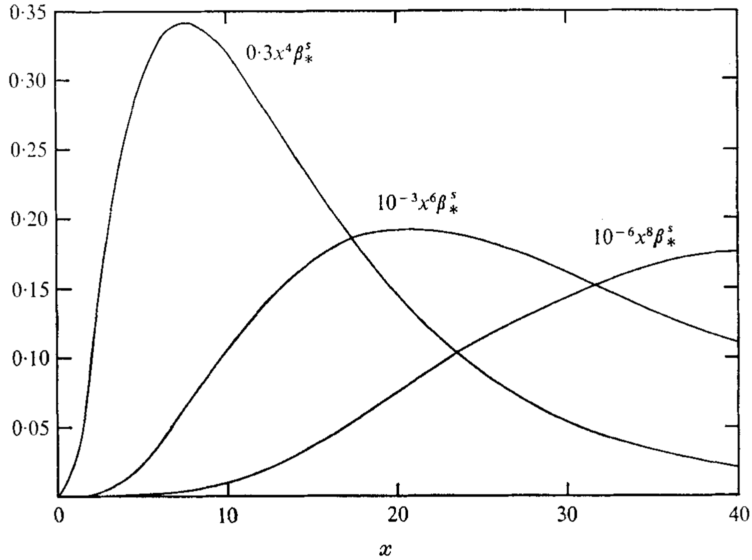

FIGURE 4. Extrapolation of figure 1 to $x = 40$, based on the curve-fitting expression (5).

The severity of these problems can be appreciated more fully if the probability density is extrapolated to larger values of x. Figure 3 shows that the curve

$$\beta^s_*(x) = a \exp(-bx^{0 \cdot 4}), \tag{5}$$

with $a = 7 \cdot 66$ and $b = 4 \cdot 45$, fits the atmospheric data ($K = 40$) quite well. By extrapolating the curves in figure 1 with the aid of (5), we obtain the results presented in figure 4. Evidently, no moments beyond the sixth could be measured with any accuracy, even if data were available (and reliable) up to $x = 40$. A similar extrapolation can be made for the laboratory data ($K = 6$); however, the atmospheric data ($K = 40$) clearly present a greater challenge, so that the discussion will be centred on these data.

Another extrapolation to larger values of x can be obtained by employing the theoretical prediction made by Oboukhov (1962) and by Gurvich & Yaglom (1967) that the probability distribution of the dissipation rate should be log-normal. The integrands $x^{2n}\beta^s_*(2n = 4, 6, 8)$ of a probability density with $K = 40$, based on a log-normal distribution of x^2, are shown in figure 5. The corresponding plot of β^s_* is given in figure 3; it is seen that a log-normal distribution of x^2 does not fit the observed distribution of x nearly as well as does the empirical curve-fitting expression (5). For example, the observed maximum of $x^4\beta^s_*$ is about $1 \cdot 1$ at $x = 7 \cdot 5$, but a log-normal distribution with $K = 40$ (figure 5) gives a maximum of $0 \cdot 83$ at $x = 6 \cdot 3$. Also, in figure 4 the maximum of $x^6\beta^s_*$ is 195 at $x = 21$, while in figure 5 that maximum is 210 at $x = 40$. Incidentally, the location of the peak of the distribution of the $2n$th moment is $x_p = K^{\frac{1}{2}(n-1)}$ if the distribution of x^2 is log-normal, so that the tenth moment, for example, would peak at $x = 1600$ if the distribution of x^2 were indeed log-normal with $K = 40$.

The poor agreement between figures 4 and 5 shows that predictions based on an assumed log-normal distribution of squared velocity derivatives cannot be

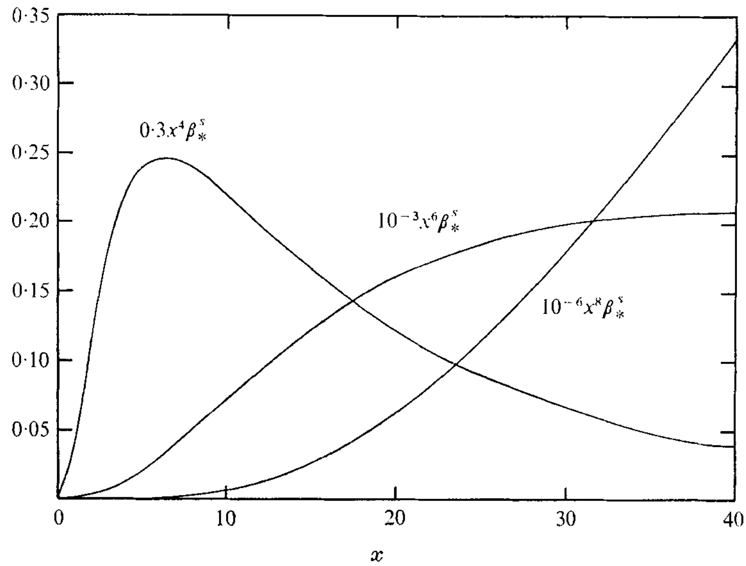

FIGURE 5. Distributions of the fourth, sixth and eighth moments of a probability distribution with $K = 40$ based on a log-normal distribution of x^2.

trusted at this stage. In this context it should be noted that Orszag (1970) has shown that the moment problem of turbulence is indeterminate if the distribution of the dissipation rate is log-normal. Therefore, we assume that β^s_* is unknown for $x > 20$, so that we cannot give analytical expressions for the maximum value of x that needs to be covered in order to obtain accurate measurements of any desired moment of a probability distribution of a velocity derivative with a certain value of K. For the time being, the accuracy of experimental results has to be checked *a posteriori*, by constructing plots such as the ones given in figures 1 and 2. This caution, however, is not always exercised in the literature: data on high order moments are often presented without substantiating evidence in the form of moment distributions. It cannot be stressed too strongly that such unsupported data should not be trusted and should not be used to check theoretical predictions.

2. Dynamic range

In the Kansas experiment (Wyngaard & Coté 1971), hot-wire signals were differentiated and low-pass filtered in one operation, using a four-pole Butterworth filter made from operational amplifiers, and recorded on a FM tape recorder. The derivative signal was subsequently played back into an analog–digital converter; the probability distribution was obtained by data processing on a digital computer. This arrangement is typical of current practice in hot-wire anemometry (see, for example, Sheih *et al.* 1971; Pierce 1972). If the derivative signal has a large kurtosis, signal excursions large compared with the standard deviation σ have to be handled faithfully by the processing equipment. We wish to establish the conditions that made such measurements possible.

The maximum value of x we need to record depends on the order of the highest moment we wish to determine and on the kurtosis of the signal; it will be denoted by

$$x_{\max} \equiv m. \tag{6}$$

Referring to figure 1, we see that m needs to be at least 20 in order to obtain a reasonably accurate value of the kurtosis of velocity derivatives in atmospheric turbulence. Also, figure 4 suggests that we would probably need $m = 50$ to determine the sixth moment. Yet higher moments require even larger values of m; it is tempting to use the curve-fitting expression (5) for 'theoretical' predictions of the value of m needed to determine any desired moment to a specified accuracy. Our attempts at such predictions did not produce any simple estimates; also, it is highly uncertain that experimental results would follow curves like (5) up to extreme values of m.

The need to cover large values of m forces one to compromise on the conventional signal-to-noise ratio of the recording equipment, so that in this kind of application tape-recorder noise tends to dominate other sources of noise. If the r.m.s. noise level is denoted by ϵ, the r.m.s. signal-to-noise ratio is given by

$$Q = \sigma/\epsilon, \tag{7}$$

where Q stands for the r.m.s. quality of the signal. For the Kansas data, $Q = 10$ approximately, which permitted the variance σ^2 to be measured with an accuracy of 1 %.

The analog tape recorder should be able to record signal excursions up to $m\sigma$ without 'clipping', 'folding' or other nonlinear distortions. If the dynamic half-range D of the recorder is used to capacity, we obtain

$$D = m\sigma/\epsilon. \tag{8}$$

Substitution of (7) into (8) yields

$$D = mQ. \tag{9}$$

This relation demonstrates one of the major trade-offs involved in measuring velocity derivatives at large Reynolds numbers. The tape recorder employed in the Kansas expedition (an Ampex FR-1300), though it had a nominal dynamic half-range of 45 db ($D = 180$), was in fact capable of 51 db ($D = 350$) without appreciable distortion (this was verified by recording sine waves of different amplitudes). With $Q = 10$ and $D = 350$ the maximum signal excursion that could be recorded thus was 35 standard deviations ($m = 35$). The number of signal excursions beyond $x = 20$ observed during several hours of recording time, however, was so small that no data beyond $x = 20$ were plotted for lack of statistical stability.

Some tape recorders do not handle overloads as well as the one used in the Kansas expedition; a more common set of numbers would be $D = 100$ (40 db), $m = 10$, $Q = 10$ (see, for example, Sheih *et al.* 1971). If one has to record on a 40 db tape recorder, and if one wants to measure signals with a kurtosis of 40 (figure 1), a reasonable compromise would be to choose $m = 20$ and $Q = 5$; this would cause a 4 % error in the determination of the variance and probably some 20 % error in the determination of K.

It would be advantageous to bypass the tape recorder if the experimental arrangements permitted, because the dynamic range of the analog differentiation circuitry is likely to be larger than that of most tape recorders. For the Kansas experiment an improvement by a factor of almost two would have resulted: at the output of the differentiator $D \simeq 620$, with $Q \simeq 18$ and $m \simeq 35$ (the maximum signal excursion of the differentiator output was matched to that of the tape-recorder input). With somewhat smaller gain, $m \simeq 60$ at $Q \simeq 10$ would have been possible, bringing a determination of the sixth moment within reach (see figure 4), were it not for prohibitively long integration times (see §4).

For signals with a lower kurtosis the situation is not nearly so desperate. Inspection of the laboratory data in figure 2 ($K = 6$) suggests that an ordinary 40 db tape recorder ($D = 100$), used in such a way that $m = 10$ and $Q = 10$, allows quite accurate measurements of the kurtosis distribution $x^4\beta_*^s$; with $m = 20$ and $Q = 5$ fair estimates of the sixth and eighth moments can be obtained if desired.

3. Digitizer resolution

The processing of the turbulence data of the Kansas expedition was performed by a digital computer; the computer processing method has now become standard. The analog signal from the tape recorder is digitized by a converter with $2R$ registers (R registers on either side of $s = 0$). If the converter is used to capacity, and if the register 'window' width is $\Delta s = \sigma \Delta x$ (recall that $s/\sigma = x$ is the normalized signal), we obtain

$$m\sigma = R\Delta s = R\sigma\Delta x, \qquad (10)$$

so that

$$\Delta x = m/R. \qquad (11)$$

Again, there is a trade-off problem: as m is increased for a given value of R, the digitizing increment Δx may become too large to allow adequate resolution of the variance distribution $x^2\beta_*^s$ (see figures 1 and 2). In the processing of the Kansas data this was no problem because a 14-bit digitizer was used ($R = 8192$) and was conservatively set at $m = 82$ (well beyond the value $m = 35$ required by the tape-recorder performance), so that $\Delta x = 0{\cdot}01$ approximately.

This issue can be defined more precisely by considering upper and lower bounds on the digitizing increment Δx. On the one hand, Δx needs to be small enough to allow good resolution of $x^2\beta_*^s$. Referring again to figure 1, we find that

$$\Delta x \leqslant 0{\cdot}2 \qquad (12)$$

would be adequate for signals with $K = 40$ (for the laboratory data of figure 2, a smaller window width, perhaps $\Delta x \leqslant 0{\cdot}1$, would be desirable).

On the other hand, the window width does not need to be so small that the noise in the analog signal causes counts to occur in adjacent registers. If we require (rather arbitrarily) that an instantaneous noise level equal to the r.m.s. value ϵ should not cause a shift to the next register (errors larger than ϵ will still do so, however), we obtain

$$\Delta s \geqslant 2\epsilon, \qquad (13)$$

that is,
$$\Delta x \geqslant 2/Q. \tag{14}$$
Combining (12) and (14), we find
$$2/Q \leqslant \Delta x \leqslant 0{\cdot}2, \tag{15}$$
and note again that the upper bound given here is a crude estimate for signals with $K = 40$ and that it should decrease with decreasing K.

Equation (15) states that, for signals with $K = 40$, the digitizer is used optimally if we choose $\Delta x = 0{\cdot}2$ and $Q = 10$. For the Kansas data, Q was indeed approximately equal to 10, but $\Delta x = 0{\cdot}01$, as we saw earlier. For this reason the counts in 12 adjacent windows were averaged, so that the effective window width was $12\Delta x = 0{\cdot}12$, which is fairly close to the (rather arbitrary) optimum derived from (15). Again, it is evident that Q has to be reasonably large in order to keep the digitizing increment small enough: (15) specifies not only an optimum value for Δx but also a lower bound on Q. For example, for $Q = 5$ there would be no point in making Δx smaller than $0{\cdot}4$; however, this would give significant resolution errors in $x^2\beta_*^{\frac{1}{2}}$ between $x = 0$ and $x = 2$ (see figure 1).

An interesting sidelight is obtained by substituting for Δx in (14) and using (9) and (11). This yields
$$D \geqslant 2R, \tag{16}$$
which states that the resolution of the digitizer does not need to be better than the quality of the tape recorder and that the data-processing system is optimized if the dynamic half-range of the tape recorder equals the number of digitizer registers. For example, an 8-bit digitizer ($2R = 256$) can be fully utilized only if one has a 48 db tape recorder. Conversely, it is evident that great improvements in analog circuitry are necessary before a 14-bit digitizer ($R = 8192$, corresponding to 84 db) is tasked to its limits.

4. Sampling time

The principal contributions to moments of high order are made at large values of x (figures 1 and 2), where the values of β_* are quite small. The time required to obtain reliable data on a moment thus increases rapidly as the order of the moment increases. If the mean value $\overline{y}$ of a signal y whose expected value is $E(y)$ is determined by analog integration over a long period of time, the mean-square relative error $\epsilon^2(\overline{y})$ is (Tennekes & Lumley 1972, §6.4)
$$\epsilon^2(\overline{y}) = \frac{\overline{\{y-E(y)\}^2}}{\{E(y)\}^2}\frac{2\tau}{T}, \tag{17}$$
where τ is the integral (time) scale of y and T is the integration time; it is assumed that the dynamic range of the instruments is adequate.

On applying (17) to the determination of the variance $\overline{s^2} = \sigma^2$ and the fourth moment $\overline{s^4} = K\sigma^4$ of a velocity derivative s with standard deviation σ, we obtain
$$\epsilon^2(\sigma^2) = 2(K-1)\,\tau_2/T_2, \tag{18}$$
$$\epsilon^2(K\sigma^4) = 2(M_8/K^2-1)\,\tau_4/T_4, \tag{19}$$
where the subscripts refer to the order of the moment (recall that $M_4 \equiv K$).

Two problems arise in connexion with equations such as (18) and (19) and their counterparts for other moments. First, the integral scales have to be estimated; second, it is unlikely that *a priori* information on the values of high order moments (e.g. M_8) is available. As far as the integral scales are concerned, the problem is complicated by the fact that the basic signal s, being a derivative, formally has a zero integral scale itself (Tennekes & Lumley 1972, §6.4). However, there is no reason to believe that powers of s have zero integral scales (Lumley 1970, §3.13). We can estimate τ_2 by assuming that the observed spectrum of s^2 (Wyngaard & Pao 1972) becomes flat as $\kappa_1 \rightarrow 0$ and taking the data at the smallest κ_1 to be representative of the value at $\kappa_1 = 0$. This is a familiar, if formally incorrect, integral-scale measurement technique (Comte-Bellot & Corrsin 1971) which in this case, because of the peculiar spectral shape, probably only gives order-of-magnitude estimates. In this way we obtain $\tau_2 = 10\,\eta/U$, where η is the Kolmogorov microscale and U is the mean velocity seen by the hot wire. A pseudo-integral scale for s itself may be estimated from the height of the maximum in its power spectrum (Lumley 1970, §3.13); that maximum is about $3\,\overline{s^2}\eta$ (see, for example, Wyngaard & Pao 1972), corresponding to an integral time scale of $3(\pi\eta/2U) \simeq 5\eta/U$. Even more crudely, we can argue that the integral length scale of s should be about 10η, because its power spectrum peaks at about $\kappa_1\eta = 0{\cdot}1$ (see Gibson *et al.* 1970; κ_1 is the streamwise component of the wavenumber vector). It appears, therefore, that a conservative estimate for τ_2 would be $10\eta/U$ (for a similar discussion, see Lumley 1970, §3.11). Estimates for $\tau_4, \tau_6, \ldots$ can be obtained from measured spectra of $s^4, s^6, \ldots$; the laboratory data of Friehe, Van Atta & Gibson (1971) suggest that τ_2, τ_4 and τ_6 all are of order $10\eta/U$. This result is not surprising: the spectra of s^2, s^4 and s^6 are quite similar, with relatively small slopes in the inertial subrange and nearly identical bandwidths. The integral scales of powers of s are thus comparable to that of s itself.

For the Kansas data, $K \simeq 40$, but no data on M_8 are available (even the extrapolation in figure 4 barely reaches the maximum of the distribution of M_8). From figure 4 an (extremely crude) estimate of M_8 may be obtained, however; the value is probably on the order of 2×10^7. With these numbers, we can estimate the integration times T_2 and T_4 for any required accuracy. For example, if $\eta = 10^{-3}$ m and $U = 5$ m/s, the time required to obtain the variance with 1 % accuracy is about 1600 s and the time required to obtain the kurtosis with 10 % accuracy is about 5000 s. One-hour runs were employed in the Kansas expedition, suggesting that the reported kurtosis values (Wyngaard & Tennekes 1970) should be fairly accurate. Note, however, that it would take 5×10^5 s (140 h!) to obtain K with 1 % accuracy.

An estimate of the time needed to obtain reasonably accurate data on M_6 would require information on M_{12}; the distribution of the latter extends to such large values of x that the extrapolation (5) cannot be trusted at all. It is, nevertheless, possible to make an alternative estimate of the required sampling time on the basis of figures 3 and 4.

We shall assume that the sampling rate used to obtain the probability distribution is chosen such that adjacent samples are approximately independent. This requires a sampling interval equal to twice the integral scale (Tennekes &

Lumley 1972, §6.4), i.e. approximately $20\eta/U$. If the number of samples in any register r is denoted by $n(r)$ and if the total number of samples is N, we obtain

$$n(r) = N\beta_* \Delta x, \tag{20}$$

assuming that $n(r) \gg 1$ and that Δx is small enough. For simplicity, let us assume that we need at least one count in the last register (the one corresponding to $x = m$). On this basis, we obtain

$$N \geqslant 1/\beta_*(m)\,\Delta x.$$

For example, in order to determine the kurtosis of the Kansas data ($K = 40$) with an accuracy of the order of 10 %, we need $m \geqslant 20$ (see figure 1), so that $\beta_*(m) \simeq \beta_*^s(m) \leqslant 3 \times 10^{-6}$ (figure 3). The effective window width employed was $\Delta x = 0{\cdot}12$; consequently, the minimum value of N required is approximately 3×10^6. The actual number of counts used was $2{\cdot}3 \times 10^7$, but the samples were not independent: the sampling interval was about 3×10^{-4} s, whereas it should have been about $2\tau = 20\eta/U = 4 \times 10^{-3}$ s (assuming $\eta = 10^{-3}$ m, $U = 5$ m/s). Roughly speaking, then, the actual samples were equivalent to about 3×10^6 independent ones, as required.

In order to measure the sixth moment with some accuracy, we would need at least $m \geqslant 40$ (see figure 4), with a corresponding $\beta_*(m) \leqslant 3 \times 10^{-8}$ (see figure 3). At an 'optimum' window width $\Delta x = 0{\cdot}2$ (corresponding to $Q = 10$; recall the discussion following (15)) this yields $N = 2 \times 10^8$. The corresponding sampling time, using independent samples at intervals of 4×10^{-3} s, would be $7{\cdot}4 \times 10^5$ s, or approximately 200 h! Even if our integral scale estimates are 10 times too large (that is, if independent samples can be obtained at 10 times the rate we estimated) it would take 20 h, still being out of practical reach.

5. Conclusions

From the point of view of an experimenter, the moment problem of the small-scale structure of turbulence is a major challenge; significant improvements will have to be made in the design and execution of experiments before reliable data on moments beyond the fourth can be obtained for velocity derivatives in geophysical turbulence. The dynamic range of most tape recorders is marginal as far as these experiments are concerned, and it appears necessary to employ on-line, real-time digitizing in order to have a chance of increasing the dynamic range to the required number of standard deviations. Even so, the integration times needed to obtain moments beyond the kurtosis remains extremely large, well beyond the range of time intervals over which reasonably stationary conditions may be expected (atmospheric turbulence records much longer than an hour or so are likely to exhibit trends). It will be necessary to obtain ensemble averages, repeating the same experiment many times until sufficient statistical stability in the extreme tails of the probability distribution is achieved. This problem is quite similar to that of aircraft in clear-air turbulence: it requires thousands of hours of recording time before one can determine the probability of encountering a severe gust with any certainty.

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